ABSTRACT

of the dissertation by Suleimenova Zoia Izteleovna
for the degree of Doctor of Philosophy (PhD)
in the educational program “8D05401-Mathematics” on the topic:

"On asymptotic behavior of solutions and issues averaging some random parabolic equations and systems of equations"

Stochastic analysis is one of the most fundamental and relevant areas of modern mathematics, both in theoretical and applied terms, which is a widely recognized fact in the scientific mathematical community. Currently, this field, based on classical probability theory and mathematical statistics, has led to the formation of new scientific areas such as actuarial, financial, and insurance mathematics, which have emerged as applied areas of these theories. Along with this, it is known that the general theory of turbulence, based on the mathematical approach proposed by A. N. Kolmogorov, also relies on the tools of stochastic analysis.

In this regard, research in the field of stochastic analysis involves not only the conceptual development and generalization of previously obtained results, but also the construction of new mathematical models, the obtaining of new results within the framework of these models, and their rigorous mathematical analysis.

Research objective. Proof of new versions of nonclassical limit theorems; proof of theorems on the representation of solutions of certain random parabolic equations in terms of conditional mathematical expectations taken along the trajectories of the corresponding stochastic processes, proof of the asymptotic normality of solutions and averaging of the heat conduction equation in a delta-correlated flow; proof of a Furstenberg-type theorem for stochastic multiplicative integrals; investigation of the evolution of a magnetic field in a linear Markov model; formulation of a discrete kinematic dynamo model, derivation of an explicit formula for the upper Lyapunov exponent within this model, and proof of the existence of a phase transition.

Object of study. Sums of independent random variables; representations of solutions to parabolic equations through conditional mathematical expectations taken along the trajectories of a random process defined by the corresponding equation, and study of solutions using these representations; incompressible linear Markov velocity flow, the magnetic induction equation in such a flow, and the Furstenberg theorem for the Markov stochastic multiplicative integral (matrix) associated with this equation; a discrete model of a kinematic dynamo and a phase transition for a magnetic field within this model.

Research methods. The methods used are characteristic functions and associated distributions of probability theory; differential equation methods of random process theory; methods of multiplicative integral theory; modern methods of stochastic differential equation theory and random process theory.

Scientific novelty. Main results presented for defense. Several variants of nonclassical limit theorems were introduced in the space of distribution functions in terms of Levy distance. Using the method of characteristic functions and the method of concomitant distributions, a universal form of generalization of the central limit theorems of Lindberg–Feller was obtained. New statements were proved that significantly generalize the classical central limit theorems.

Parabolic equations and their systems were rewritten in the form of equations whose infinitesimal operators are determined by random processes. The characteristics of the solutions were obtained in terms of conditional mathematical expectations. Using Laplace transform, it was proven that the solutions of some equations reduce to solutions of simple ordinary differential equations with constant coefficients. The asymptotic normality of the solutions of a parabolic equation perturbed by a random process of the "white noise" type was established. In the problem of averaging random parabolic equations, parabolic equations have been derived for the average temperature field and for the two-point average temperature field.

An incompressible linear Markov model is introduced, and a special multiplicative stochastic integral (matrix) is constructed for the evolution of the magnetic field. Based on L. Hörmander's theory, a Furstenberg-type theorem is proved for a parabolic operator and random processes with strictly positive transition probability density. A discrete model of the kinematic dynamo is proposed and its probabilistic characteristics are investigated.

Explicit formulas for Lyapunov exponents and magnetic field moments have been obtained. Regions of exponential growth and decay of magnetic field energy corresponding to the phase transition phenomenon have been determined. New stochastic models and explicit calculation formulas describing the dynamics and stability of the magnetic field have been developed, and the application of Furstenberg-type theorems to magnetic field evolution problems has been demonstrated.

Approbation. The results of the dissertation were presented and discussed at the following international conferences: "Third International Conference on Analysis and Applied Mathematics" (Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan, ICAAM 2016); "VI Congress of the Turkic World Mathematical Society" (2–5 October 2017, Astana, Kazakhstan, TWMS 2017); "Current Problems of Pure and Applied Mathematics" (Almaty, Kazakhstan, 22–25 June 2017); "Applied Mathematics, Control and Informatics" (3–5 October 2012,

Belgorod, Russia); “Actual Problems of Stochastic Analysis” (20–21 April 2021, Tashkent, Uzbekistan); “TWMS 2017. VI Congress of the Turkic World Mathematical Society” (2–5 October 2017, Astana, Kazakhstan); “Theoretical and Applied Problems of Mathematics, Mechanics, and Informatics” (12–13 June 2019, Karaganda, Kazakhstan); “Limit Theorems of Probability Theory and Mathematical Statistics” (26–28 September 2022, Tashkent, Uzbekistan); as well as at scientific seminars of the Department of Mathematics and the Department of Computational Science and Statistics of the Faculty of Mechanics and Mathematics, Al-Farabi Kazakh National University.

Publications. Based on the results of the dissertation research, 17 works were published, including:

– 3 articles in scientific journals included in the second and third quartiles (Q2, Q3) of Clarivate Analytics Journal Citation Reports, respectively, and/or with CiteScore 99, 68, 56, 7 percentiles in the Scopus database, respectively:

1. *N.Akanbay, S.A.Molchanov, Z.I. Suleimenova.* Theorem of Furstenberg type for multiplicative stochastic integrals. - Random operatore and Stochastic equations., 28:3 (2020), 163-176. (DOI:10.1515/rose-2020-2035, (Scopus percentile– 61, SJR= 0.164, Scopus quartile – Q4)

2. *N. Akanbay., Z.I.Suleimenova., S.K.Tapeyeva.* The multiplicative integral and the evolution of the magnetic field in the Marcov linear model - Journal of Mathematics, Mechanics and Computer Science, 2025, Volume 126, issue 2, Pages 25-40. (DOI: 10.26577/jmmcs2025126202), (Scopus percentile– 61, SJR= 0.272, Scopus quartile – Q3) Тексер!

3. *Н.Аканбай., С.А.Молчанов., З.И.Сулейменова.* Модель кинематического динамо – Успехи математических наук, 79:4 (478), 2024, 171-172. *N.Akanbay., S.A.Molchanov Z.I.Suleimenova.* Model of discrete kinematic dynamo- Russian Mathematical Surveys, 2024, Volume 79, Issue 4, Pages 724–726. (DOI:10.4213/rm10190), (Scopus percentile– 62, SJR= 0.684, Scopus quartile – Q2)

– 6 articles in journals recommended by the Committee for Quality Assurance in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan

1. Akanbay E.N., Suleymenova Z.I. *On moment equations of the magnetic field in random Markov linear flows.* Vestnik KazGU. Series: Mathematics, Mechanics, Informatics, No. 12, 1998, pp. 192–198. MRNTI: 27.35.29.

2. N.Akanbay, A.B.Akhmedov, Suleymenova Z.I. *On certain variants of the non-classical central limit theorem.* Vestnik KazNU, Series: Mathematics, Mechanics, Informatics, 2013, No. 2 (77), pp. 52–64. MRNTI: 27.43.15.

3. N. Akanbay, Suleymenova Z.I. *On the probabilistic origins of some combinatorial relations.* Vestnik KazNPU named after Abai, Series: Physical-Mathematical Sciences, No. 2 (50), 2015, pp. 14–20.

4. Suleymenova Z.I. *On the probabilistic representation of the solution of the Cauchy problem for a parabolic equation and its applications.* Vestnik of the Kyrgyz-Russian Slavic University, 2016, Vol. 16, No. 9, pp. 25–31.

5. Akanbay N., Suleymenova Z.I. *On the probabilistic solution of the Cauchy problem for parabolic equations.* Al-Farabi Kazakh National University Journal of Mathematics, Mechanics and Computer Science, No. 2 (94), 2017, pp. 21–34.

6. Akanbay E.N., Suleymenova Z.I., Tapeeva S.K. *Asymptotic normality of the solution of a parabolic equation with rapidly oscillating random coefficients.* Vestnik KazNITU, 2020, No. 6 (142), pp. 572–581.

– 8 publications in the collection of abstracts of international conferences.

1. Akanbay N., Suleymenova Z., Tapeeva S. *On asymptotic normality of solutions of the Cauchy problem of a parabolic equation with random right side.* ICAAM 2016. Third International Conference on Analysis and Applied Mathematics. Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan. The Abstract Book, p. 201.
2. Akanbay N., Nurkhanova M.S., Suleimenova Z.I. *On averaging of the equation of thermal conductivity in a random delta-correlated flow.* TWMS 2017. VI Congress of the Turkic World Mathematical Society, 2–5 October 2017, Astana, Kazakhstan. The Abstract Book, pp. 287–288.
3. Akanbay N., Suleymenova Z.I. *On the asymptotic behavior of solutions of certain random parabolic equations.* Proceedings of the International Conference "Current Problems of Pure and Applied Mathematics," Almaty, 22–25 august 2017, pp. 87–88.
4. Akanbay N., Suleymenova Z.I., Tapeeva S.K. *On moment equations of the temperature field in a random flow with instantaneous time correlation.* Proceedings of the International Scientific Conference "Applied Mathematics, Control and Informatics," 3–5 October 2012, Belgorod, Russia, Vol. I, pp. 16–19.
5. Akanbay N., Suleymenova Z.I., Tapeeva S.K. *On averaging the temperature field equation in a delta-correlated in time random flow.* Proceedings of the Scientific Conference "Actual Problems of Stochastic Analysis," 20–21 February 2021, Tashkent, Uzbekistan, pp. 113–116.
6. Akanbay N., Nurkhanova M.S., Suleimenova Z.I. *Linear model of the hydromagnetic dynamo and multiplicative products of Markov random matrices.* TWMS 2017. VI Congress of the Turkic World Mathematical Society, 2–5 October 2017, Astana, Kazakhstan. The Abstract Book, pp. 286–287.
7. Akanbay N., Suleymenova Z. I., Tapeeva S.K. *Evolution of the magnetic field in a Markov linear model.* Proceedings of the International Scientific Conference "Theoretical and Applied Problems of Mathematics, Mechanics, and Informatics," 12–13 June 2019, Karaganda, Kazakhstan, pp. 61–62.
8. Akanbay N., Suleimenova Z.I., Tapeyeva S.K. *Multiplicative integral and asymptotic analysis of the magnetic field equation in the Markov linear model.* Abstracts of Communications, International Conference "Limit Theorems of Probability Theory and Mathematical Statistics," 26–28 September 2022, Tashkent, Uzbekistan, pp. 8–10.

Structure of the dissertation. The dissertation consists of normative references, an introduction, three main sections (each of which contains subsections), a conclusion, and a list of references.

The introduction provides justification for the relevance of the research, the purpose of the work, mathematical formulation of the problems under study, the results to be defended, the object of the study, the scientific novelty of the results obtained and their theoretical and practical significance, as well as information about the testing of the results, the author's publications, the structure of the dissertation, and a brief overview of the content of the sections and the conclusion.

In the first chapter of the dissertation, using the method of characteristic functions and the method of concomitant distributions, several variants of non-classical limit theorems of probability theory are proved, i.e., limit theorems that do not require the condition of uniform infinitesimal smallness of the terms, formulated in the space of distribution functions in terms of the Lévy distance. It is shown that the obtained theorems are essential generalizations of the central limit theorems of Lindberg–Feller.

The second chapter of the dissertation proves theorems stating that solutions to a series of parabolic equations and systems of equations can be represented as conditional mathematical expectations along the trajectories of random processes constructed as solutions to the corresponding stochastic differential equations, whose infinitesimal operators coincide with the operators on the right-hand sides of the equations under consideration.

The results obtained are used to prove statements about the possibility of obtaining parabolic equations for the joint characteristic function of various functionals of random process trajectories, their stochastic integrals, and Ito stochastic integrals from these processes. In several particular cases, it is shown that after Laplace transformation, the solutions of such equations reduce to solutions of ordinary differential equations with constant coefficients, and corresponding examples are given. In addition, the asymptotic normality of the solution of a parabolic equation perturbed by a white noise random process is proved.

Issues of averaging random parabolic equations are considered using the example of the heat conduction equation in a delta-correlated random flow, and the corresponding parabolic equations are derived for the average temperature field and for the two-point average temperature field at the same moment in time.

The third section of the dissertation is devoted to the study of the magnetic induction equation in an incompressible flow in a kinematic setting, i.e., in a model in which the probabilistic-statistical characteristics of the velocity field are assumed to be known and the reverse influence of the magnetic field on the velocity flow is not taken into account.

The concept of an incompressible linear Markov model is introduced, which in a certain sense generalizes incompressible velocity flows with short correlation and Markov structure, and for the magnetic field within this model, an explicit integral formula is obtained, expressing it through a special multiplicative stochastic integral.

In section 3.1 of the third chapter, taking into account that the existence of a strictly positive density of transition probabilities of a random process with an infinitesimal operator, which is a degenerate parabolic operator containing mixed derivatives with respect to arguments, follows from L. Hörmander theory of hypoelliptic second-order differential equations, a Furstenberg-type theorem is proved for the stochastic multiplicative integral (matrix), i.e., the multiplicative product of special Markov matrices associated with an incompressible linear Markov model. We briefly consider the application of this theorem to the study of magnetic field evolution.

Section 3.2 of the third chapter is devoted to the study of the magnetic induction equation in three-dimensional space. It introduces the concept of a discrete model of a kinematic dynamo, provides its probabilistic description, and investigates similar issues for the case when the velocity field in this discrete model is a delta-correlated Gaussian field. In particular, first, the process whose infinitesimal operator coincides with the operator on the right-hand side of the magnetic induction equation is defined as the solution of the corresponding stochastic differential equation (Lagrange trajectory), after which the solution of the magnetic induction equation is represented using a discrete analogue of the Feynman–Kac formula. The study of magnetic field moments and corresponding Lyapunov indices is based on this representation.

The main results obtained in section 3.2 are as follows:

- It has been proven that the largest Lyapunov exponent is strictly positive ($\gamma_2 > 0$) (Furstenberg-type theorem);
- An explicit formula for calculating the exponent γ_2 has been obtained;
- It has been shown that the odd moments of the magnetic field are zero, and for the even moments, in particular for the second moment, exact formulas for calculation have been found;
- Theorems on the existence of a phase transition for the exponent γ_2, corresponding to the existence of regions of exponential growth and attenuation of the magnetic field energy, have been proved, and explicit descriptions of these regions have been given.

The conclusion presents the main results and conclusions obtained in the course of the dissertation work.

Degree of implementation. The results obtained in the dissertation are applied in the use of central limit theorems of probability theory, when recording solutions to random parabolic equations through conditional mathematical expectations along the trajectories of processes corresponding to these equations, as well as when researching solutions based on such representations and when determining the asymptotic behavior of ordinary and multiplicative products of random matrices.

Scope of application. The results obtained are used in solving problems in mathematics, hydrodynamics, and physics, as well as in probability theory, characteristic function theory, random process theory, multiplicative integral theory, and stochastic differential equation theory.

Candidate's contribution to the article. Based on the results of the dissertation, three articles were published in journals included in the Scopus and Web of Science databases, five articles in domestic journals, and one article in a Kyrgyzstan journal. The candidate has thoroughly mastered the methods and approaches used, as well as the theoretical material presented in the articles, carefully analyzing and comprehending it. At the same time, each result obtained in the sections of the work is directly related to the candidate's activities. In some cases requiring additional theoretical knowledge, the candidate consulted with domestic and foreign scientific supervisors.